

Autocorrelations and thermal fragility of anyonic loops in topologically quantum ordered systems

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Are systems that display topological quantum order (TQO), and have a gap to excitations, hardware fault-tolerant at finite temperatures? We show that in models that display low d -dimensional gaugelike symmetries, such as Kitaev's and its generalizations, the expectation value of topological symmetry operators vanishes at *any* nonzero temperature, a phenomenon that we coined *thermal fragility*. The autocorrelation time for the nonlocal topological quantities in these systems may remain finite even in the thermodynamic limit. We provide explicit expressions for the autocorrelation functions in Kitaev's toric code model. If temperatures far below the gap may be achieved then these autocorrelation times, albeit finite, can be made large. The physical engine behind the loss of correlations at large spatial and/or temporal distance is the proliferation of topological defects at any finite temperature as a result of a *dimensional reduction*. This raises an important question: How may we best quantify the degree of protection of quantum information in a topologically ordered system at finite temperature?

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I. INTRODUCTION

The perseverance of information over long times in the simplest of memory devices is related to the existence of large associated autocorrelation times. The storage of information is intimately tied to the breaking of ergodicity at scales much smaller than the autocorrelation time. Classical information can be reliably stored in magnetically or in electrically (permanently) polarized materials. From the physicist's perspective, this reliability may be directly linked to the existence of an *order parameter* (its *macroscopic* magnetization or polarization) which characterizes a *collective* and robust property of the material below an ordering transition temperature. At its core, nonergodicity implies the existence of a *generalized order parameter* (e.g., the overlap parameter of spin glasses).

The reliable storage of quantum information is a real challenge. The uncontrolled interactions between a quantum system and its environment or measurement apparatus introduce noise (errors) in the system leading to decoherence of pure quantum superposition states. Fortunately, quantum states can, in principle, be encoded fault tolerantly and be *protected* against decoherence, thus preventing loss of information.¹ This idea lies at the heart of topological quantum order (TQO) systems as first advanced by Kitaev.² Assuming that errors are of a *local* nature, topological quantum memories (e.g., surface codes²) seem to be intrinsically stable because of *physical* fault tolerance to weak quasilocal perturbations. However, are these quantum memories robust to thermal effects?

In this work, we analyze the effect of temperature on zero-temperature ($T=0$) topologically ordered quantum systems,^{3,4} such as Kitaev's toric code² and honeycomb models⁵ and generalizations thereof. To this end, we need to present two concepts that were introduced in our previous work.⁶ One is the concept of finite- T TQO, and the other of rank- n TQO. In that same work we studied the thermal fragility of topological operators in $D=2$ lattice models. Our

results⁶ concerning the singular character of the $T=0$ TQO in one notable system (Kitaev's toric code model) have later been reaffirmed in work by Castelnovo and Chamon⁷ in their study of the topological entanglement entropy. In the present work we will present extensions of our ideas to higher spatial dimensions D and expand on the physical reasons leading to thermal fragility. In particular, we show that a general \mathbb{Z}_k gauge theory in D spatial dimensions in a system with periodic boundary conditions displays rank- $n=k^D$ TQO. Nevertheless, although a thermodynamic phase transition may occur, the system is thermally fragile. We investigate not only the thermodynamic but also the dynamical aspects of thermal fragility, and in cases such as Kitaev's toric code model we also obtain exact analytic time-dependent results thanks to our duality mappings.⁶

II. LANDAU ORDERS VS TQO

Before defining TQO, and to put this latter concept in perspective, let us briefly review the rudiments of a Landau order parameter. The Landau order parameter, a macroscopic property measuring the degree of order in a state of matter, is customarily associated with the breaking of a global symmetry. Thus the existence of an order parameter is directly attached to the phenomenon of spontaneous symmetry breaking (SSB). This concept, that involves an infinite number of degrees of freedom, is so fundamental to condensed matter and particle physics that many excellent textbooks (see, for example, Ref. 8) have spent entire chapters (or even a full book⁹) describing it. For the present purposes, we illustrate the concept in the simple case of a ferromagnet. A piece of iron at high temperatures it is in a *disordered* paramagnetic phase. Below a certain temperature T_c the system *orders*, i.e., it magnetizes, and with the appearance of the order parameter (magnetization) there is a breaking of the rotational symmetry.¹⁰ In the (ferro)magnetic phase there is a net magnetization \mathbf{M} that persists all the way to zero temperature (where it attains its maximal value). The magnetization can,

mathematically, be written as a linear combination of *quasilocal* operators V (e.g., V is the local spin operator in a Heisenberg model). The main point to stress here is that the operator V may distinguish between different ground states (GSs) $|g_\alpha\rangle$ and $|g_\beta\rangle$ of the material

$$\langle g_\alpha | V | g_\alpha \rangle \neq \langle g_\beta | V | g_\beta \rangle, \quad (1)$$

and equivalently at finite temperatures,

$$\langle V \rangle_\alpha \neq \langle V \rangle_\beta. \quad (2)$$

Here and throughout, the angular brackets refer to a thermodynamic average: For any quantity, say \mathcal{A} , we have that $\langle \hat{\mathcal{A}} \rangle = \text{tr}[\rho \hat{\mathcal{A}}]$. The density matrix $\rho = \mathcal{Z}^{-1} \exp[-\beta H]$ with $\mathcal{Z} = \text{tr}[\exp[-\beta H]]$ where the inverse temperature $\beta = 1/(k_B T)$ with k_B the Boltzmann constant. Similarly, we define a (trace-class) density matrix $\rho_\alpha = \mathcal{Z}_\alpha^{-1} \exp[-\beta H_\alpha]$ (with $\mathcal{Z}_\alpha = \text{tr}[\exp[-\beta H_\alpha]]$ the partition function) corresponding to the Hamiltonian H endowed with terms which favor order in the state $|g_\alpha\rangle$. These subscripts (α) are those appearing in the thermodynamic averages of Eq. (2). A particular realization of ρ_α for the problems that will interest us will be given below [Eq. (11)].

Symmetry plays a key role in dictating the fundamental properties of matter. Symmetry generally implies the existence of conserved charges with unique physical consequences. Most Landau orders are inherently tied to broken global symmetries. There are symmetries other than global. For instance, the symmetries in gauge theories are local; such local symmetries cannot be broken.¹¹ Recently, a general classification (and their physical consequences) of these and other types of symmetries was proposed.⁶ A symmetry is termed a d -dimensional gaugelike symmetry (d -GLS) if the minimal nonempty set of symmetry operations operate on a d -dimensional spatial volume.^{6,12} Thus global symmetries—those of the usual Landau-type—correspond to $d=D$ (here the symmetry operators act nontrivially on the entire D -dimensional system), and gauge symmetries (which are local in nature) correspond to $d=0$ (as the symmetry operators act nontrivially only on quasilocal [or ($d=0$)-dimensional] regions). General symmetries may lie anywhere in between these two extremes: $0 \leq d \leq D$. The groups associated with such symmetries can be denoted as \mathcal{G}_d ,^{6,12} and may be Abelian or non-Abelian. (In the following we will only consider unitary representations.) Similar to global orders that cannot appear in low-dimensional systems, SSB of $d=0,1$ discrete or $d \leq 2$ continuous GLSs is precluded. The engine behind the absence of SSB of these symmetries are the very same topological defects which inhibit symmetry breaking in low-dimensional systems (e.g., domain walls that restore discrete $d \leq 1$ GLSs). In Refs. 6 we have presented several examples that show how topological defects may inhibit SSB in systems with $d \leq 1$ discrete GLSs. The statements that we will make below pertain to general systems, both in the continuum and on lattices. For explicit forms, in what follows, we will often provide expressions and refer to systems defined on D -dimensional hypercubic lattices of size $L \times L \times \cdots \times L$.

Topological quantum order is an order of matter that lies beyond the Landau symmetry-breaking description. TQO cannot be described by local order parameters, and can only be described by nonlocal quantum numbers. Proposed candidates for the latter are topological sensitivity of the GS degeneracy, quasiparticle fractional statistics, edge states, topological entanglement entropy, etc. As we showed elsewhere,⁶ these different quantum numbers may give rise to different definitions that do not always overlap. A theory may satisfy some of these conditions but not others.

For the purposes of this work, and since we are interested in a concept that cannot be described by the Landau paradigm, we will now define TQO by extending on Kitaev's original condition² for zero temperature TQO. Given a set of n orthonormal GSs $\{|g_\alpha\rangle\}_{\alpha=1,\dots,n}$, with $1 < n \leq N_g$ where N_g is the total number of GSs of a given Hamiltonian H , $T=0$ rank- n TQO exists if and only if for any bounded operator V with compact support (i.e., any *quasilocal operator* V),

$$\langle g_\alpha | V | g_\beta \rangle = v \delta_{\alpha\beta} + c_{\alpha\beta}, \quad (3)$$

where v is a constant and $c_{\alpha\beta}$ is a correction that vanishes in the thermodynamic limit. This is indeed a condition on *non-distinguishability* of GSs through local measurements. Here and throughout, we will employ Greek letters α and β to denote orthogonal states in the GS manifold. Note that Eq. (3) applies only to systems with degenerate GSs. Following standard conventions, we will keep employing β as a shorthand for the inverse temperature $1/(k_B T)$. A finite- T ($T > 0$) generalization of Eq. (3) is provided by the following *condition for TQO*:⁶

$$\langle V \rangle_\alpha \equiv \text{tr}[\rho_\alpha V] = v + c_{\alpha\alpha}(L), \quad \forall \alpha, \quad (4)$$

with $c_{\alpha\alpha}$ a correction that tends to zero in the thermodynamic limit. TQO systems⁶ satisfy *both* the $T=0$ condition of Eq. (3) as well as the finite- T condition of Eq. (4). In Ref. 6, we illustrated how TQO can be readily achieved in systems harboring $d \leq 1$ discrete GLSs and $d \leq 2$ continuous GLSs.

We will now discuss general *quantum error detection conditions* in such TQO systems. These conditions allow us to assess whether errors can be detected in TQO systems. It is important to emphasize that systems displaying finite- T TQO [obeying the conditions of Eqs. (3) and (4)], do not necessarily satisfy the quantum error detection conditions below. The zero temperature error detection condition is given by

$$[\hat{P}_0 V \hat{P}_0, \hat{T}_\mu] = P_0 [V, \hat{T}_\mu] P_0 = 0, \quad (5)$$

where $\hat{P}_0 = \sum_\alpha |g_\alpha\rangle\langle g_\alpha|$ is the protected subspace, and \hat{T}_μ 's represent the logical operators which are not part of the *code's stabilizer*.¹³ These operators are nontrivial symmetries of H , i.e.,

$$[H, \hat{T}_\mu] = 0, \quad (6)$$

and encode the braiding operations that ensure topological degeneracy of the GS manifold. We will reserve the use of the Greek indices μ and ν to the operators $\{\hat{T}_\mu\}$. It is important to emphasize that the $T=0$ TQO quantum error detection

condition, Eq. (5), applies for systems with degenerate and nondegenerate GSs [unlike Eq. (3) which only applies to systems with degenerate GSs].

Clearly, when $c_{\alpha\beta}=0$ condition (3) implies Eq. (5). To see this, we write the commutator of Eq. (5) longhand to find that

$$\sum_{\alpha,\beta} \langle g_\alpha | V | g_\beta \rangle [| g_\alpha \rangle \langle g_\beta |, \hat{T}_\mu] = v [\hat{P}_0, \hat{T}_\mu] = 0 \quad (7)$$

identically (regardless of the specific symmetry operator \hat{T}_μ). This follows from Eq. (6) whenever Eq. (3) holds with $c_{\alpha\beta}=0$. Thus condition (3) is sufficient (but not necessary) to ensure the general error detection condition of Eq. (5).

Motivated by the zero-temperature condition above, one can propose the following obvious extension of Eq. (5) to finite temperatures¹⁴

$$[\rho^{1/2} V \rho^{1/2}, \hat{T}_\mu] = \rho^{1/2} [V, \hat{T}_\mu] \rho^{1/2} = 0, \quad (8)$$

for all quasilocal operators V . Here, the propagation of a local error at finite temperatures (V) causes no harm to the logical operators $\{\hat{T}_\mu\}$. In the “typical finite-temperature subspace,” the local errors ($\rho^{1/2} V \rho^{1/2}$) do not alter the algebra of the operators $\{\hat{T}_\mu\}$. In anyonic computing schemes, the algebra of such nonlocal operators—the algebra of $\{\hat{T}_\mu\}$ —is what enables quantum memories. Similarly, the algebra of related nonlocal operators encodes the braiding operators that may perform topological quantum computing. In Appendix A, we will show that the quantum error detection condition of Eq. (8) generally cannot be satisfied for any finite-size system nor, more generally, in any other system which does not display a finite- T transition. The existence of a finite- T phase transition is a necessary but not sufficient condition for the quantum error detection condition of Eq. (8) to hold.

We will now briefly relate a weaker version of the finite- T detection condition of Eq. (8) to our earlier finite- T TQO condition of Eq. (4). This will suggest that although finite- T transitions (nonanalyticities in the free energy) are mandated to ensure finite-temperature error detection, these transitions cannot be accompanied by SSB. Thus these transitions may be more akin to those in gauge theories. To this end, we note that if Eq. (8) holds then, in particular, the finite- T expectation value

$$\langle V \rangle = \langle \hat{T}_\mu V \hat{T}_\mu^\dagger \rangle. \quad (9)$$

We now consider an extension of Eq. (9) that is valid in the thermodynamic limit for *all* quasilocal V

$$\langle V \rangle_\alpha = \langle \hat{T}_\mu V \hat{T}_\mu^\dagger \rangle_\alpha, \quad (10)$$

stating that there is no SSB of the symmetries spanned by $\{\hat{T}_\mu\}$ —at least insofar as any local observable V can detect. It is worth emphasizing that in Eq. (10), the indices α and μ generally need not be the same. The absence of SSB detectable by local observables is the physical content of Eq. (4) with

$$\rho_\alpha = Z_\alpha^{-1} \exp[-\beta(H + h_\alpha \hat{T}_\alpha)], \quad (11)$$

where $h_\alpha \rightarrow 0$.

In our recent work,⁶ we further developed a symmetry principle for TQO. We related certain symmetry transformations of a system to the existence of TQO as defined by Eqs. (3) and (4). We emphasized the fundamental role d -GLSSs play in establishing that order and disallowing SSB of local quantities. We basically proved sufficient symmetry conditions for a system to be topologically quantum ordered: *When in a gapped system of finite interaction range and strength, the GSs (each of which can be chosen by the application of an infinitesimal field) may be linked by discrete $d \leq 1$ or by continuous $d \leq 2$ GLSSs $U \in \mathcal{G}_d$, then a system that satisfies the $T=0$ conditions of Eq. (3) exhibits finite- T TQO [in the sense of Eqs. (3) and (4)].* We refer the reader to Ref. 6 for a comprehensive explanation. The quantum error detection conditions of Eq. (8) are far more restrictive than the finite- T TQO conditions regarding the robustness of the system to all quasilocal perturbations V .

Although it is possible in many cases⁶ to satisfy the weaker version of Eq. (8) [that is Eq. (10)] by the use of d -GLSSs, unless they are biased by hand, the logical operators $\{\hat{T}_\mu\}$ always have a vanishing expectation value at any long-time equilibrium finite- T state (i.e., any possible Gibbs state) in a system with finite range interactions

$$\langle \hat{T}_\mu \rangle_\alpha = 0. \quad (12)$$

A related, more practical, consequence is that $\{\hat{T}_\mu\}$ may generally exhibit finite autocorrelation times. That is, at all positive temperatures,

$$G_{\hat{T}_\mu}(t) = \langle \hat{T}_\mu(0) \hat{T}_\mu(t) \rangle \quad (13)$$

with $G_{\hat{T}_\mu}(t) \rightarrow 0$ as $(|t|/\tau) \rightarrow \infty$; there is an inherently finite autocorrelation time τ , whose size is limited by thermal fluctuations (but not by system size). This autocorrelation time remains finite even in the thermodynamic limit. An indefinitely self-correcting TQO surface code can only exist at exactly zero temperature.

In Sec. III A we will show how this is explicitly realized in the Kitaev's toric code model [Eqs. (39), (40), and (43)]. Before embarking on an analysis of specific cases, let us first analyze general relations.

In several specific cases, such as Kitaev's toric code model, which we will analyze below, it is possible to find a new representation in which an initial local Hamiltonian remains local yet the nonlocal topological anyonic loops become objects of low effective dimensionality. For instance, in Kitaev's toric code model, the nonlocal toric cycle loops become point ($d=0$) fields. In such cases of low effective dimensionality, the absence of finite temperature SSB in low dimensions guarantees that Eq. (12) holds even without performing more detailed expansions or bounds.

Below, we prove this result for the particular case of a symmetry operator which is independent not only of the code's stabilizer but also of the Hamiltonian itself. Here, a

transformation exists which turns the symmetry \hat{T}_μ into a ($d=0$)-dimensional operator. Similar to Eq. (11), we may define

$$\mathcal{Z}(h_\mu) = \text{tr}[\exp[-\beta(H - h_\mu \hat{T}_\mu)]] \quad (14)$$

with no summation over repeated indices (μ) implicit. If the logical operator is independent of the argument of a local Hamiltonian $H = \sum_i H_i$ (here $\{H_i\}$ are local operators), then the partition function of Eq. (14) simply factorizes

$$\mathcal{Z}(h_\mu) = \mathcal{Z} \times z(h_\mu). \quad (15)$$

From Eq. (15), the expectation value

$$\langle \hat{T}_\mu \rangle = \lim_{h_\mu \rightarrow 0^+} \frac{1}{\beta \mathcal{Z}(h_\mu)} \frac{\partial}{\partial h_\mu} \mathcal{Z}(h_\mu) = \lim_{h_\mu \rightarrow 0^+} \frac{1}{\beta z(h_\mu)} \frac{\partial}{\partial h_\mu} z(h_\mu) = 0. \quad (16)$$

The expectation value evaluated with $z(h_\mu)$ that encompasses only one site cannot exhibit SSB [in formal terms, $z(h_\mu)$ is the partition function of a ($d=0$)-dimensional system]. Thus, quite universally, the expectation value of any such logical operator vanishes. We emphasize that this holds for all systems (both in the thermodynamic limit and finite size systems). All that matters in the derivation of Eqs. (15) and (16) is that \hat{T}_μ is independent of the variables $\{H_i\}$ that are added to form the code's Hamiltonian. In Sec. III we will work out these and related expectation values in detail for Kitaev's toric code model.

Equation (12) may be extended more generally to *nonlocal* high-dimensional operators (\hat{R}_a) that need not (i) lie outside the code's stabilizer or (ii) be symmetries of H . For instance, in lattice gauge theories, \hat{R}_a can pertain to a Wilson loop of a divergent perimeter.¹⁵ By performing a low temperature series expansion about the ordered state, one generally finds that, similar to Eq. (12), all nonlocal operators $\{\hat{R}_a\}$ of dimension $d > 0$ have a vanishing expectation value at finite temperatures,

$$\langle \hat{R}_a \rangle = 0. \quad (17)$$

Equation (17) follows from an asymptotic perimeter law type bound (see, e.g.,¹⁶)

$$|\langle \hat{R}_a \rangle| \leq A e^{-cm}, \quad (18)$$

with $m = \mathcal{O}(L^d)$ the number of local fields that lie in the support of \hat{R}_a and A, c positive constants.¹⁷ As seen from Eq. (18), for $m \rightarrow \infty$ (as befits any nonlocal \hat{R}_a), this expectation value vanishes. Hand in hand,

$$\langle \hat{R}_a(0) \hat{R}_a(t) \rangle_{|t| \rightarrow \infty} \rightarrow 0. \quad (19)$$

Related results [for both the commutator in Eq. (8) as well as the commutator between topological quantities] are afforded by simple extensions (carried in the Appendixes) of the Lieb-Robinson bounds known to apply for local quantities in spin systems with local interactions.¹⁸

Consider next $d=0$ symmetries, i.e., operators $\{\hat{T}_\mu\}$ that span only a zero-dimensional volume (or a finite number of points on a lattice). For the operators $\{\hat{T}_\mu\}$ to realize a non-trivial ray representation which leads to a topological degeneracy, these local symmetry operators cannot commute with one another. Elitzur's theorem¹¹ states that any quantity which is not invariant under all local symmetries must vanish at any finite temperature. Thus, for any quasilocal \hat{T}_μ symmetry operator (including all operators which may be defined on any finite size lattice), there can never be a SSB of \hat{T}_μ and once again Eq. (12) follows. Considerations similar to those of Eq. (16) can be enacted.¹⁹ Couched in the language more commonly used by researchers in anyonic quantum computing, we can say that in this case—the case of general local ($d=0$) operators (which includes any *finite* lattice as a special realization of $d=0$ symmetries which are here enforced by the limited physical extent $D=0$ of the system)—if we are given a set of symmetry group operators $\{\hat{T}_\mu\}$, i.e., $[H, \hat{T}_\mu] = 0$, such that they form a non-Abelian group or an Abelian group with a ray (nonvector) representation, then we can prove that $\langle \hat{T}_\mu \rangle = 0$. For example, in the case of a ray representation $\hat{T}_\mu \hat{T}_\nu = e^{i\phi_{\mu\nu}} \hat{T}_\nu \hat{T}_\mu$, $\langle \hat{T}_\mu \rangle = \text{tr}[\hat{T}_\mu \rho \hat{T}_\mu^\dagger] = e^{i\phi_{\mu\mu}} \langle \hat{T}_\mu \rangle = 0$. From the dimension of the irreducible representation we infer the degeneracy of the GS subspace. (We remind the reader that for a continuous connected group of symmetries all finite-dimensional ray representations are equivalent to vector representations.)

III. THERMAL FRAGILITY: WORKED-OUT EXAMPLES

In earlier work,² the presence of a gap in the energy spectrum together with the existence of TQO as defined above [Eq. (3) with $c_{\alpha\beta} = 0$] were suggested to be sufficient to guarantee the protection of quantum information. The physical intuition behind this was that properties of the protected (GS) subspace are stable with respect to weak local perturbations. How does temperature affect this conclusion? Is there any other additional requirement needed for protection? We just mentioned that the simple generalization of the quantum error detection condition, Eq. (8), may generally fail at all temperatures $T \neq 0$. It is commonly believed that the existence of a finite gap between the ground and first excited states protects properties associated with $T=0$ TQO up to a finite energy scale $k_B T$ smaller than the gap Δ since thermal fluctuations are suppressed by the Boltzmann factor $\exp[-\Delta/k_B T]$. Our results⁶ showed that this assumption is, in general, incorrect for long times. We originally coined the term *thermal fragility* to describe this state of affairs.⁶ We showed that in some prominent TQO models, the entropic weight associated with defects out weights their Boltzmann penalty of $\exp[-\Delta/k_B T]$ at any finite temperature: The equilibrium states are always disordered. In what follows, we expand on these concepts and consider extensions of previous examples to higher spatial dimensions D . In Sec. III A, we derive the autocorrelation function for Kitaev's toric code model. We then review (in Sec. III B) a new high-dimensional extension of Kitaev's toric code model and

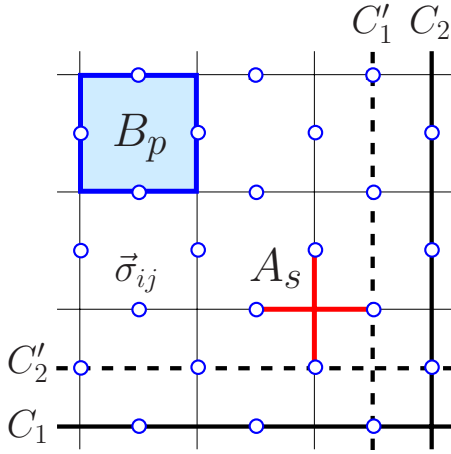


FIG. 1. (Color online) Elementary (cell) plaquette B_p and star A_s interaction terms in Kitaev's toric code model. Hollow circles in the bonds (links) represent an $S=1/2$ degree of freedom, while thick (dashed or solid) lines represent the topological symmetry operators (see text).

prove that this system displays TQO and a finite autocorrelation time. Finally, in Sec. III C we discuss the situation for Kitaev's honeycomb model.

A. $D=2$ Kitaev's toric code model

For the sake of clarity and because of historical importance we start by summarizing the main findings of Ref. 6 regarding Kitaev's toric code model² in $D=2$. The model is defined on a square lattice with $L \times L = N_s$ sites, where on each bond (or link) (ij) it is defined a $S=1/2$ degree of freedom indicated by a Pauli matrix $\vec{\sigma}_{ij}$, thus defining a Hilbert space of dimension 2^{2N_s} (see Fig. 1). The model Hamiltonian is given by

$$H_K = - \sum_s A_s - \sum_p B_p \quad (20)$$

with Hermitian operators (whose eigenvalues are ± 1)

$$A_s = \prod_{(ij) \in \text{star}(s)} \sigma_{ij}^x, \quad B_p = \prod_{(ij) \in \text{plaquette}(p)} \sigma_{ij}^z, \quad (21)$$

and σ_{ij}^κ ($\kappa=x, y, z$) representing Pauli matrices. B_p and A_s describe the plaquette (or face) and star (or vertex) operators, respectively, with $(\forall s, s', p, p')$

$$[A_s, A_{s'}] = [B_p, B_{p'}] = [A_s, B_p] = 0, \quad (22)$$

thus generating an Abelian group called *code's stabilizer*.² In the presence of periodic boundary conditions, the plaquette and star operators satisfy the constraint

$$\prod_s A_s = \prod_p B_p = 1, \quad (23)$$

and the two $d=1$ \mathbb{Z}_2 symmetries are given by²

$$\begin{aligned} Z_{1,2} &= \prod_{(ij) \in C_{1,2}} \sigma_{ij}^z, \quad X_{1,2} = \prod_{(ij) \in C'_{1,2}} \sigma_{ij}^x, \\ \{X_\mu, Z_\mu\} &= 0, [X_\mu, Z_\nu] = 0, \mu \neq \nu, \end{aligned} \quad (24)$$

where $C_1(C'_2)$ are horizontal and $C_2(C'_1)$ vertical closed contours [i.e., loops on the lattice (dual lattice)]. The logical operators $Z_{1,2}$ and $X_{1,2}$ commute with the code's stabilizer but are not part of it, thus acting nontrivially on the two *encoded* toric code qubits.

As shown in Ref. 6 H_K is related to Wen's plaquette model⁴ and to two Ising chains by *exact* duality mappings. Therefore, these three models share the *same* spectrum. The GS (protected subspace of the code) is four-fold degenerate (Abelian $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry) and there is a gap to excitations. The spectrum is basically that of two uncoupled circular Ising chains ($2N_s$ is the total number of links of the original $D=2$ lattice)

$$\tilde{H}_K = - \sum_{p=1}^{N_s} \sigma_p^z \sigma_{p+1}^z - \sum_{s=1}^{N_s} \sigma_s^x \sigma_{s+1}^x, \quad (25)$$

with GS energy $E_0 = -2N_s$ and a gap to the first excited state equals to 4 (this value of 4 and not 2 arises as in the presence of periodic boundary conditions, only an even number of domain walls are possible). This example clearly illustrates the fact that TQO is a property of states and not of the Hamiltonian spectrum.⁶

The elementary excitations of H_K are of two types²

$$\begin{aligned} |\Psi_z(\Gamma)\rangle &= \prod_{(ij) \in \Gamma} \sigma_{ij}^z |\Psi_0\rangle \equiv S^z(\Gamma) |\Psi_0\rangle, \\ |\Psi_x(\Gamma')\rangle &= \prod_{(ij) \in \Gamma'} \sigma_{ij}^x |\Psi_0\rangle \equiv S^x(\Gamma') |\Psi_0\rangle, \end{aligned} \quad (26)$$

where $\Gamma(\Gamma')$ is an open string on the lattice (dual lattice) and $|\Psi_0\rangle$ is a GS. [If $\Gamma(\Gamma')$ would be closed contours which circumscribe an entire toric cycle then the string operators $S^{x,z}$ would become the toric symmetries of Eq. (24).] In the case of the open contours of Eq. (26), the operators $S^{x,z}$ generate excitations at the end points of these strings (thus always coming in pairs) with Abelian fractional statistics (anyons). Excitations living on the vertices represent *electric charges* while the ones living on the plaquettes represent *magnetic vortices*. These magnetic and electric type excitations obey fusion rules that enable Abelian quantum computation. Due to the exact equivalence between Kitaev's model and the Ising chains, no nontrivial finite temperature SSB or other transitions can take place. The spectrum exhibits a multitude of low-energy states. At any finite temperature, no matter how small, entropic contributions to the free energy overwhelm energy penalties and lead to a free energy which is everywhere analytic.⁶

The operators of Eqs. (26) are not symmetries of H and, for divergent loop size, are specific examples of the nonlocal operators $\{\hat{R}_a\}$ that we considered earlier [see, e.g., Eq. (18)]. In the one-dimensional Ising duality mapping of Eq. (25), we may represent the string operators of Eq. (26) as the creation operators for domain walls in the $D=1$ Ising model. When Γ and Γ' intersect at any even number of bonds, a representation is

$$S^z(\Gamma) = \prod_{s_1 < s \leq s_2} \sigma_s^z, \quad S^x(\Gamma') = \prod_{p_1 < p \leq p_2} \sigma_p^x. \quad (27)$$

In Eq. (27), s_1 and s_2 denote the end points of the string Γ of Eq. (26). Similarly, $p_{1,2}$ are the plaquettes which form the endpoints of the string Γ' on the dual lattice. When Γ and Γ' share an odd number of bonds, we can represent the string operators as follows:

$$S^z(\Gamma) = \sigma_{p_2}^z \prod_{s_1 < s \leq s_2} \sigma_s^z, \quad S^x(\Gamma') = \prod_{p_1 < p \leq p_2} \sigma_p^x. \quad (28)$$

For any finite lattice and $T > 0$

$$\begin{aligned} \langle Z_\mu \rangle &= \frac{\text{tr}[\exp[-\beta H_K] Z_\mu]}{\text{tr}[\exp[-\beta H_K]]} \\ &= \frac{\text{tr}[X_\mu \exp[-\beta H_K] X_\mu Z_\mu]}{\text{tr}[\exp[-\beta H_K]]} = \frac{\text{tr}[\exp[-\beta H_K] X_\mu Z_\mu X_\mu]}{\text{tr}[\exp[-\beta H_K]]} \\ &= -\langle Z_\mu \rangle = 0, \end{aligned} \quad (29)$$

where we have used the property $[H_K, X_\mu] = 0 = [H_K, Z_\mu]$ and the cyclic invariance of the trace tr , which is performed over the eigenstates of σ_{ij}^z with eigenvalues ± 1 . Similarly, $\langle X_\mu \rangle = 0$. Indeed, this is a special case of the more general argument depicted in Sec. II.

This does not, in principle, preclude SSB in the thermodynamic limit. One needs to restrict the configurations over which the trace is performed. In the thermodynamic limit, derivatives of the partition function and associated free energy need not be analytic single-valued functions (when SSB occurs the expectation values depend on how the limit $\vec{h} \rightarrow 0$ is taken). To this end, let us define the (generating) partition function of the model with the constraint (23)

$$\begin{aligned} \mathcal{Z} &= \text{tr} \left[\exp \left[-\beta \left(H_K - \sum_{\mu=1,2} (h_{x,\mu} X_\mu + h_{z,\mu} Z_\mu) \right) \right] \right] \\ &= [(2 \cosh \beta)^{N_s} + (2 \sinh \beta)^{N_s}]^2 \cosh \beta h_1 \cosh \beta h_2, \end{aligned} \quad (30)$$

where $h_\mu = \sqrt{h_{x,\mu}^2 + h_{z,\mu}^2}$. Thus the free energy per bond, $F = -(\beta^{-1}/2N_s) \ln \mathcal{Z}[h_{\kappa,\mu}=0]$, is analytic for all finite β , and displays a singularity at $T=0$ (inherited from the Ising chain). That means that no finite- T phase transition occurs in Kitaev's model. Moreover, from Eq. (31), we can compute the expectation values of the topological operators with the result

$$\begin{aligned} \langle Z_\mu \rangle &= \lim_{h_{z,\mu} \rightarrow 0^+} \frac{\partial}{\partial (\beta h_{z,\mu})} \ln \mathcal{Z} = \lim_{h_{z,\mu} \rightarrow 0^+} \frac{h_{z,\mu}}{h_\mu} \tanh(\beta h_\mu), \\ \langle X_\mu \rangle &= \lim_{h_{x,\mu} \rightarrow 0^+} \frac{\partial}{\partial (\beta h_{x,\mu})} \ln \mathcal{Z} = \lim_{h_{x,\mu} \rightarrow 0^+} \frac{h_{x,\mu}}{h_\mu} \tanh(\beta h_\mu), \\ \langle Z_1 \rangle &= \langle Z_2 \rangle = \langle X_1 \rangle = \langle X_2 \rangle = 0. \end{aligned} \quad (31)$$

This indicates that *the existence of a gap in this system may not protect a finite expectation value of the toric code operators $X_{1,2}$ or $Z_{1,2}$ for any finite temperature $T > 0$.*^{6,20} At $T=0$ these expectation values are finite and equal to unity, reflecting the nonanalyticity of F at $T=0$. The physical reason behind this result is the proliferation of topological defects (solitons) at any finite T . The Boltzmann suppression

becomes ineffective at sufficiently long times and this might be bad news for a robust quantum memory.²¹ In the presence of additional fields [see last terms in Eq. (31)], with the two component vector $\vec{h}_\mu = (h_{x,\mu}, h_{z,\mu})$ at site μ , we may define the two component vector

$$\hat{n}_\mu = \frac{1}{|\vec{h}_\mu|} (h_{x,\mu}, h_{z,\mu}) \quad (32)$$

and set $([H_K - \sum_{\mu=1,2} (h_{x,\mu} X_\mu + h_{z,\mu} Z_\mu), Q_\mu] = 0)$

$$\begin{aligned} Q_1 &= X_1 n_{x1} + Z_1 n_{z1}, \\ Q_2 &= X_2 n_{x2} + Z_2 n_{z2} \end{aligned} \quad (33)$$

to be the counterparts of two single Ising spins σ_{sc}^x and σ_{pd}^z which are located at site numbers c and d of the two respective Ising chains (that of the s and that of the p varieties). c and d can be chosen to be any integers such that $1 \leq c, d \leq N_s$. These two spins [along with the spins appearing in Eq. (25) satisfy precisely the same algebra and set of constraints as the original variables in Kitaev's model [Eqs. (20), (21), and (24)]. The vanishing expectation values of Eq. (31) for both the finite and infinite system can be understood as the statement that $\langle \sigma_{sc}^z \rangle = \langle \sigma_{pd}^z \rangle = 0$ for any one-dimensional Ising system when the single on-site magnetic fields $\vec{h}_{c,d} \rightarrow 0$.

The entropy associated with $d=1$ type domain walls is logarithmic in the system size N_s . By contrast, the energy penalty for these domain walls is finite and size independent. As a result, for sufficiently large systems, entropic gains will outweigh energy penalties. In particular, in the thermodynamic limit, there is no SSB of the $d=1$ GLSs at any temperature $T > 0$. In Ref. 6, we showed how all static correlation functions may be computed via our mapping to the Ising chain.

We can now address the dynamical aspects of thermal fragility in Kitaev's toric code model. From its mapping to two uncoupled Ising chains, we can immediately determine the time autocorrelation functions of the toric code operators. As we have shown, Kitaev's toric code operators $\{X_{1,2}, Z_{1,2}\}$ map into single spins in an Ising chain. Thus, we can employ the results obtained in Refs. 22 and 23 concerning autocorrelation of spins in Ising chains in a system with Glauber-type dynamics. In the Appendixes [Eqs. (D3) and (D4)], we will outline standard master equations which are used to determine the dynamics. When these equations depend only on the system's spectra (and not the precise real-space form of the GSs) then we can use our mapping of Eq. (33) to relate the dynamics of the nonlocal topological quantities of Kitaev's toric code model to single spin dynamics in Ising chains. We find that

$$G_{X_\mu}(t) \equiv \langle X_\mu(0) X_\mu(t) \rangle \quad (34)$$

obeys, for $t > 0$, the relation

$$\frac{\partial G_{X_\mu}}{\partial t} = \chi \frac{\tanh^2 \beta - 1}{\tanh^2 \beta + 1} I_0 [t \tanh \beta \tanh 2\beta] e^{-\chi t}. \quad (35)$$

Here, χ is a constant setting the time scale for the evolution of the system and I_0 is the modified Bessel function. For low temperatures, at short times,²³

$$|t| \ll \chi^{-1}, \quad (36)$$

the autocorrelation is given by

$$G_{X_\mu; \text{short-time}}(t) \simeq e^{-\sqrt{2[1-\tanh 2\beta]}\chi|t|}. \quad (37)$$

At intermediate times,²³

$$\chi^{-1} \ll |t| \ll \chi^{-1} \frac{1}{1 - \tanh 2\beta}, \quad (38)$$

the autocorrelation of the topological string operators is well approximated by a Cole-Davidson form²⁴ in the frequency domain and stretched exponential in time,

$$G_{X_\mu; \text{intermediate-time}}(t) \simeq e^{-\sqrt{|t|/\tau}}, \quad (39)$$

with the equilibration time

$$\tau = \frac{\pi}{4\chi(1 - \tanh 2\beta)}. \quad (40)$$

The constant τ is independent of the system size (for large systems) and is finite for all temperatures $T > 0$. If the temperature can be made much smaller than the gap, i.e., if $\beta \gg 1$, then τ , albeit being finite, can be made large ($\tau \simeq [\pi/(8\chi)\exp[4\beta]]$). We recall that in Kitaev's toric code model on the torus (the system with periodic boundary conditions) the gap between the GS and lowest excited energy levels is equal to four: $\Delta=4$ [see the discussion after Eq. (25)]. [Similarly, the gap for the model on an open surface (open boundary conditions) is given by $\Delta=2$.] Thus, for Kitaev's toric code model on a torus, in the limit of small temperatures the inverse Boltzmann factor $\exp[\beta\Delta]$ scales in the same fashion as the equilibration time τ . The important feature of the scaling of the autocorrelation time in the Kitaev's model is that, for large systems, it is system size independent at all temperatures. Similarly, in the long-time limit,

$$|t| \gg \chi^{-1} \frac{1}{1 - \tanh 2\beta}, \quad (41)$$

the autocorrelation function is well approximated by

$$G_{X_\mu; \text{long-time}}(t) = \sqrt{\frac{1 + \tanh 2\beta}{2\pi\chi|t|(1 - \tanh 2\beta)}} e^{-\chi(1 - \tanh 2\beta)|t|}. \quad (42)$$

Identical relations hold for the autocorrelators $G_{Z_\mu}(t) \equiv \langle Z_\mu(0)Z_\mu(t) \rangle$. Thus, as we increase the system size the error rate will always be finite at any $T > 0$. In other words, any topological quantum memory at finite (nonzero) temperatures might not sustain self-correction indefinitely.

We now discuss Eq. (8). It is readily seen that this condition is *violated* here. For instance, if we choose $V = \sigma_{ij}^x$, and $\hat{T}_\mu = Z_1$ with $(ij) \in C_1$ we have, for any finite β , that

$$\rho^{1/2}[\sigma_{ij}^x, Z_1]\rho^{1/2} \neq 0, \quad (43)$$

as $\{\sigma_{ij}^x, \sigma_{ij}^z\} = 0$. This is, of course, a particular realization of

the general result of Appendix A. *Kitaev's toric code model has a $T=0$ transition and thus cannot satisfy the condition of Appendix A.*

On the other hand, by choosing $V = \sigma_{ij}^x$, which has the property that

$$(1 + B_p)\sigma_{ij}^x(1 + B_p) = 0, \quad (44)$$

and realizing that the GS projection operator is given by

$$\hat{P}_0 = \prod_s \left(\frac{1 + A_s}{2} \right) \prod_p \left(\frac{1 + B_p}{2} \right), \quad (45)$$

the $T=0$ detection condition, Eq. (5), is trivially satisfied. Similarly, if one chooses $V = \sigma_{ij}^z$ since

$$(1 + A_s)\sigma_{ij}^z(1 + A_s) = 0, \quad (46)$$

and, in general, for Kitaev's model Eq. (5) is satisfied. That the $T=0$ conditions of Eq. (5) are satisfied also follows from the fact that Eq. (3) implies that Eq. (5), and as we proved in Ref. 6 Kitaev's model, satisfies the $T=0$ conditions of Eq. (3).

B. Generalizations of Kitaev's Toric code model

There is obviously no connection between the lack of thermodynamic phase transition (as in Kitaev's model) and the existence of TQO as defined in the Introduction. Indeed, we will now explore two models that have TQO yet display finite- T phase transitions as signaled by nonanalyticities in the free energy F . The first model we will consider is a $D=3$ extension of Kitaev's model and the second a $D=3$ Z_k gauge theory. In both of these systems, no SSB occurs (and TQO is indeed materialized). Nevertheless, the system's free energy displays nonanalyticities at finite temperatures.

The $D=3$ Kitaev's extended model (KE3D) in a cubic lattice has a Hamiltonian formally written as Eq. (20) with star (or vertex) operators A_s , each comprising the six nearest-neighbors to a site, and planar plaquette operators B_p , each involving 4 spins (see Fig. 2). This Hamiltonian is basically a $D=3$ Ising gauge theory ($-\sum_p B_p$) augmented by the sum of all local symmetry generators ($-\sum_s A_s$). The constraints of Eq. (23) get replaced by

$$\prod_s A_s = 1, \quad \prod_{p \in \text{cube}} B_p = 1, \quad (47)$$

where *cube* includes the six plaquettes which form the cube. Since no constraint couples the vertex and plaquette operators, the partition function is simply

$$Z_{3D} = Z_{3D \text{ Ising gauge}} \times Z_{1D \text{ Ising}}. \quad (48)$$

Let us first show that KE3D does show a thermodynamic phase transition at $T > 0$. From Eq. (48) the free energy is given by $F_{3D}(\beta) = F_{3D \text{ Ising gauge}}(\beta) + F_{1D \text{ Ising}}(\beta)$, where $F_{3D \text{ Ising gauge}}(\beta) = F_{3D \text{ Ising}}(\beta^*)$ with the dual β^* satisfying $\sinh 2\beta \sinh 2\beta^* = 1$. Clearly, the free energy of KE3D displays a $T=0$ nonanalyticity coming from $F_{1D \text{ Ising}}(\beta)$ and a finite- T nonanalyticity resulting from $F_{3D \text{ Ising gauge}}(\beta)$ at $\beta_c = 0.7614$ 23.

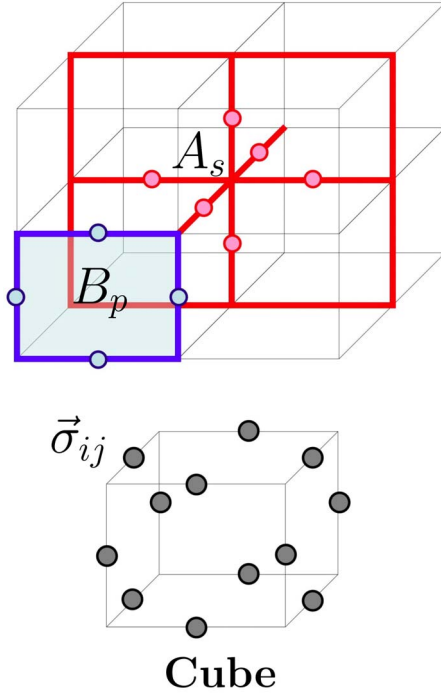


FIG. 2. (Color online) $D=3$ extension of Kitaev's Toric code model. On each bond (ij) , there is an $S=1/2$ degree of freedom indicated by a Pauli matrix $\vec{\sigma}_{ij}$. Notice that elementary plaquettes are planar while stars involve six neighbors in the x , y , and z space directions. A *cube* involves six plaquettes.

We now show that the $D=3$ Ising gauge theory displays (at least) rank $n=8$ TQO. (See the definition of rank- n TQO given in the Introduction.) Let us start by writing eight GSs of the $D=3$ Ising gauge theory on a cubic lattice of size L^3 which is endowed with periodic boundary conditions

$$\begin{aligned} |g_{+++}\rangle &= \mathcal{N} \sum_{c \in \{+++ \}} |c\rangle, \\ |g_{++-}\rangle &= \mathcal{N} \sum_{c \in \{++- \}} |c\rangle, \\ &\vdots \\ |g_{---}\rangle &= \mathcal{N} \sum_{c \in \{--- \}} |c\rangle. \end{aligned} \quad (49)$$

In Eqs. (49), the states $\{|c\rangle\}$ span all states in the σ_{ij}^z basis which (i) have $B_p=1$ for each plaquette p [see the definition of B_p in Eq. (21)] and (ii) lie in a specific topological sector and \mathcal{N} is a uniform normalization constant. The eight topological sectors $[(+++), (++-), \dots, (---)]$ are labeled by three Toric invariants $(\mu=1, 2, 3)$

$$Z_\mu = \prod_{(ij) \in C_\mu} \sigma_{ij}^z. \quad (50)$$

Each of the $\otimes_{(ij)} \sigma_{ij}^z$ eigenstates $|c\rangle$ is an eigenstate of the three toric operators $\{Z_\mu\}_{\mu=1}^3$. In Eq. (50), each of the three cycles $C_{1,2,3}$ is a path which circumscribes one *toric* cycle (e.g., a cycle along each of the three cubic axes). Each of the

states in Eq. (49) transforms as a singlet under all of the cubic lattice star operations

$$A_s = \prod_{i \in \text{star}(s)} \sigma_{si}^x, \quad (51)$$

with the product above performed over all six bonds (si) which have the vertex s as one of their end points. This is so as the star operations of Eq. (51) link states $|c\rangle$ within the same topological sector. That is, if

$$A_s |c\rangle = |c'\rangle \quad (52)$$

then, as $[Z_\mu, A_s]=0$ for all μ and s ,

$$Z_\mu |c\rangle = Z_\mu |c'\rangle. \quad (53)$$

Given the invariance of the eight GSs of Eq. (49) (and thus of any superpositions thereof) under all of the $d=0$ symmetries $\{A_s\}$, we may proceed to demonstrate TQO. To this end, let us decompose any quasilocal operator V into the component invariant under *all* of the $d=0$ GLSs of Eq. (51) (labeled by $V_{A;0}$) and component $V_{A;\perp}$ which does not transform as a singlet under all of these operations: $V = V_{A;0} + V_{A;\perp}$. For any state $|g\rangle$ which lies in the eight-dimensional space spanned by the states of Eq. (49), and which transforms as a singlet under all $d=0$ GLSs \mathcal{G} , we have

$$\langle g | V_{A;\perp} | g \rangle = 0. \quad (54)$$

All that we need to consider are thus the local ($d=0$) symmetry invariant components $V_{A;0}$ of the quasilocal operator V . The quasilocal operators invariant under all of the symmetries of Eq. (51) are built out of product of a finite number of operators $\{A_s\}$ and plaquette operators $\{B_p\}$ [see Eq. (21)]. We must now show that any such quasilocal operator $V_{A;0}$ attains the same expectation value in each of the states of Eq. (49). To this end, we consider the following three *connecting operators*:

$$X_\mu = \prod_{(ij) \perp \hat{e}_\mu} \sigma_{ij}^x. \quad (55)$$

These operators are the $D=3$ extension of the operators $X_{1,2}$ of the $D=2$ \mathbb{Z}_2 gauge theory [Eq. (24)].

In Eq. (55), the product is taken over all bonds (ij) which lie in planes perpendicular to the cubic direction \hat{e}_μ . These operators commute with one another and are symmetries of the Ising gauge Hamiltonian: $[H, X_\mu]=0$. Acting with a particular X_μ on any state (e.g., any *vortexless* σ^z eigenbasis state $|c\rangle$) which is an eigenstate of all $\{B_p\}$ operators with unit eigenvalue leads to states which are eigenvectors of $\{B_p\}$ with unit eigenvalue. More generally, for any operators A_s and B_p , we have

$$[A_s, X_\mu] = [B_p, X_\mu] = 0. \quad (56)$$

As a consequence of Eq. (56) and the fact that all quasilocal operators $V_{A;0}$ are multinomials in $\{A_s, B_p\}$, we have that

$$[V_{A;0}, X_\mu] = 0. \quad (57)$$

Moreover, these operators satisfy the following algebra with respect to the toric symmetries $\{Z_\mu\}$:

$$\{X_\mu, Z_\mu\} = 0, \quad [X_\mu, Z_\nu] = 0, \quad \mu \neq \nu. \quad (58)$$

As a consequence of Eq. (58), we see that the eight states of Eq. (49) are related to one another by these operators. For instance,

$$\begin{aligned} |g_{+++}\rangle &= X_3 |g_{+++}\rangle, \\ |g_{--+}\rangle &= X_1 X_3 |g_{+++}\rangle, \end{aligned} \quad (59)$$

etc.. Therefore, the eight $d=1$ GLS operators $\Pi_\mu X_\mu^{n_\mu}$ with $n_\mu=0,1$ form a group \mathcal{G} (of a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ character). These operators suffice to link all of the states of Eq. (49) with one another. By unitarity, these generators also link any set of eight orthogonal states in the space spanned by Eq. (49).

By Eqs. (54) and (57), the expectation value of any quasilocal operator V is the same in all GSs spanned by the $n=8$ GSs of Eq. (49). Thus the $D=3$ Ising gauge theory exhibits (at least) $(\text{rank}-n)=8$ TQO.

We note, in passing, that considerations similar to those above may be enacted for general \mathbb{Z}_k gauge theories ($k=2,3,4,\dots$) on a hypercubic D -dimensional lattice. This theory is defined by

$$H_{\mathbb{Z}_k} = -\frac{1}{2} \sum_p (U_{ij} U_{jk} U_{kl} U_{li} + \text{H.c.}) \quad (60)$$

where, on every link (ij) , a parallel transporter

$$\begin{aligned} U_{ij} &= e^{i\theta_{ij}}, \\ \theta_{ij} &= 2\pi n_{ij}/k \end{aligned} \quad (61)$$

with n_{ij} an arbitrary integer. The elements U_{ij} of Eq. (61) satisfy a \mathbb{Z}_k algebra. Here, instead of Eq. (55), we set

$$\hat{T}_\mu = \prod_{(ij) \perp \hat{e}_\mu} e^{i(2\pi/k)L_{ij}^z}. \quad (62)$$

In Eq. (62), L_{ij}^z is the generator of rotation of the \mathbb{Z}_k variable on bond (ij) . Replicating the proof given above, we find that the \mathbb{Z}_k gauge theory on the D -dimensional lattice exhibits (at least) $\text{rank}-n=k^D$ TQO.

It remains to be proven that at any finite $T>0$ no topological symmetry operator may acquire a nonvanishing expectation value. For example,

$$\langle Z_{C_\mu} \rangle = \left\langle \prod_{(ij) \in C_\mu} \sigma_{ij}^z \right\rangle = 0, \quad (63)$$

for $\{C_\mu\}$ loops around the toric cycles. [Equation (63) is a particular realization of Eq. (12).] Due to the decoupling of the plaquette $\{B_p\}$ and vertex $\{A_s\}$ operators, the expectation value $\langle Z_{C_1} Z_{C_2} \rangle$ is given by its value for a classical $D=3$ Ising gauge theory. However, as seen by large and small coupling expansions,¹⁶ the correlator $\langle Z_{C_1} Z_{C_2} \rangle = \langle \prod_{(ij) \in C_1} \sigma_{ij}^z \prod_{(ij) \in C_2} \sigma_{ij}^z \rangle$ vanishes in the $D=3$ Ising gauge theory as the bounding contours $C_{1,2}$ are taken to be infinite. [For finite size systems, $\langle Z_{C_\mu} \rangle$ vanishes as no SSB is possible.] In (i) the *confined* phase [$\beta < \beta_{c2}$] this correlator vanishes exponentially in the area of the minimal surface R bounded by the toric cycles $C_{1,2}$ while in (ii) the *deconfined*

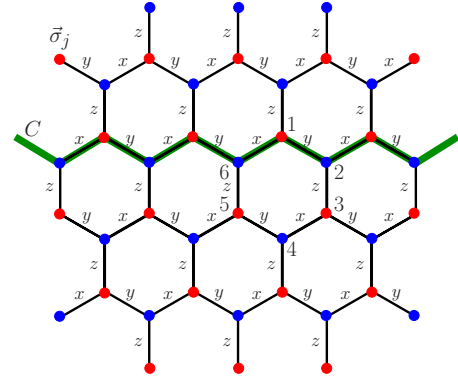


FIG. 3. (Color online) Kitaev's model on a honeycomb lattice and three types of bonds. On each vertex there is an $S=1/2$ degree of freedom indicated by a Pauli matrix $\vec{\sigma}_j$. C represents an arbitrary contour drawn on the lattice.

phase [$\beta > \beta_{c2}$], the pair correlator $\langle Z_{P_1} Z_{P_2} \rangle$ vanishes exponentially in the total length of the contours C_1 and C_2 . In the limit of far separated contours C_1 and C_2 , both (i) and (ii) reaffirm Eq. (63). Equation (63) is suggestive of a finite autocorrelation time at all positive temperatures.

C. Kitaev's honeycomb model

Kitaev's model on the honeycomb lattice⁵ is defined by the following $S=1/2$ Hamiltonian (Fig. 3)

$$H_{K_h} = -J_x \sum_{x \text{ bonds}} \sigma_j^x \sigma_k^x - J_y \sum_{y \text{ bonds}} \sigma_j^y \sigma_k^y - J_z \sum_{z \text{ bonds}} \sigma_j^z \sigma_k^z. \quad (64)$$

Here, we find that Elitzur's theorem mandates that *all* nonvanishing expectation values must be of the form²⁵

$$\hat{O}_C = \prod_{j \in C} \sigma_j^\gamma, \quad (65)$$

with C any contour (set of contours) drawn on the lattice and γ is the bond direction which is orthogonal to the path C . Let us consider embedding this system on a torus with g handles (a torus of genus g). For closed contours C which do not span an entire toric cycle, \hat{O}_C is not independent of the code's stabilizer. In fact, there is a rather simple connection between these operators and the local anyon charge which measures the number of local high energy defects. For a closed contour C which is an elementary closed hexagonal loop, we have \hat{O}_h which is the anyon charge associated with a given hexagon h . Rather specifically,

$$\hat{O}_h = \sigma_1^z \sigma_2^x \sigma_3^y \sigma_4^z \sigma_5^x \sigma_6^y \quad (66)$$

with 1–6 labeling the vertices of any given hexagon of Fig. 3. The spin polarization directions μ for all spins σ_i^μ in the product of Eq. (66) have been chosen to correspond to the single bond direction μ (x, y , or z) that is attached to the site i and does not lie on the hexagonal path.

As shown by Kitaev, within the GS sector, $\langle \hat{O}_h \rangle = 1$ for all h . Extending the arguments of the strong and weak coupling expansions of gauge theories,¹⁶ we now find that the cor-

relator between any two contours $\langle \hat{O}_C \hat{O}_{C'} \rangle$ scales as (i) $e^{-c_1 A}$ at high temperatures with A the area bounded by C and C' and c_1 a positive constant and, if an ordered phase exists, scales at low temperatures as (ii) $e^{-c_2(|C|+|C'|)}$ with $|C|+|C'|$ the total perimeter of the two loops and c_2 another positive constant. Thus, for cycles C which span the entire lattice we find as before that no $d=1$ loops of the form of Eq. (65) can attain a finite expectation value.

Let us now consider closed contours $C_{1,2}$ which span the same toric cycle. For a system with periodic boundary conditions, at $T=0$, we now have for closed contours $C_{1,2}$,

$$\left\langle \prod_{i \in C_1} \sigma_i^z \prod_{j \in C_2} \sigma_j^z \right\rangle = \left\langle \prod_{h \in D} \hat{O}_h \right\rangle = 1. \quad (67)$$

Here, D is the domain bounded by two contours $C_{1,2}$ which go around an entire cycle of the torus (see, e.g., the thick solid line in Fig. 3). Taking the separation between the two contours C_1 and C_2 to be very large, we have that, at $T=0$,

$$\left\langle \prod_{i \in C_1} \sigma_i^z \prod_{j \in C_2} \sigma_j^z \right\rangle \rightarrow \left\langle \prod_{i \in C_1} \sigma_i^z \right\rangle \left\langle \prod_{j \in C_2} \sigma_j^z \right\rangle. \quad (68)$$

This implies that, at $T=0$,

$$\left\langle \prod_{i \in C_1} \sigma_i^z \right\rangle = \pm 1. \quad (69)$$

At $T>0$, no SSB may occur [also in the thermodynamic limit, see Eq. (18)] and all quantities of the form Eq. (69) must be zero.

We now write down a formal finite temperature solution to the partition function and comment on low- and high- T series expansions. Using fermionization,²⁵ Kitaev's honeycomb model can be cast as a model for fermions $\{d_r\}$ on a square lattice with a site-dependent chemical potential $\eta_r = \pm 1$.

$$\begin{aligned} H_{K_h} = & J_x \sum_r (d_r^\dagger + d_r)(d_{r+\hat{e}_x}^\dagger - d_{r+\hat{e}_x}) + J_y \sum_r (d_r^\dagger + d_r)(d_{r+\hat{e}_y}^\dagger \\ & - d_{r+\hat{e}_y}) + J_z \sum_r \eta_r (2d_r^\dagger d_r - 1) \\ \equiv & d^\dagger M_\eta d + d^\dagger W_\eta \tilde{d}^\dagger + \tilde{d} W_\eta^\dagger d. \end{aligned} \quad (70)$$

The fermionization of Ref. 25 has been recently invoked to attain very interesting results in extensions of Kitaev's honeycomb model.²⁶ In Eq. (70), $\{\vec{r}\}$ denote the centers of the vertical bonds. The unit vector \hat{e}_y connects two z bonds and crosses a y bond; see Fig. 3. A similar definition holds for \hat{e}_x .²⁵ The form of Eq. (70) is very similar to the Fermi representation of the $D=2$ Ising model. In the last line of Eq. (70),

$$d^\dagger = (\{d_r^\dagger\}), \quad \tilde{d} = (\{d_r\}). \quad (71)$$

With $\Lambda = (d^\dagger, \tilde{d})^T$ and

$$N_\eta = \begin{pmatrix} M_\eta & 2W_\eta \\ 2W_\eta^\dagger & -M_\eta \end{pmatrix}, \quad (72)$$

the partition function for the Fermi bilinear of Eq. (70) is easy to write down

$$Z_{K_h} = \sum_{\{\eta_r\}} e^{-(\beta/2) \text{tr}[M_\eta]} |\det(e^{\beta N_\eta} + 1)|^{1/2}. \quad (73)$$

Within the GS sector (the one with no anyons), the system is translationally invariant and its spectrum⁵ is that of a p -wave type BCS pairing problem.^{25,26}

Due to the sum over all of the $2^{N_s/2}$ Ising configurations $\{\eta_r\}$, the complete partition function of Eq. (73) is nontrivial. The existence of a transition as temperature is varied is not as immediate as in Kitaev's toric code model. Translational invariance appears only for uniform $\eta_r = 1$ or -1 for all r and by a simple unitary transformation to a system in which η_r is constant on entire horizontal lines.²⁵ Nevertheless, bounds on correlators are easily established: Any fermionic correlator computed with the full partition function of Eq. (73) is bounded from above by its value when computed within the sector $\{\eta_r\}$ which maximizes its value. The finite T fixed sector correlators for the quadratic Hamiltonian of Eq. (70) can be computed with the aid of Wick's theorem. If the GS sector (that with $\eta_r = 1$ everywhere) is gapped then at all temperatures, the correlation functions exhibit exponentially decaying correlations.

We may expand Eq. (73) in low- and high- T series. A high- T series may be derived by an expansion of $|\det(e^{\beta N_\eta} + 1)|$ in powers of β . At low temperatures, the GS terms correspond to the phase $\eta_r = 1$ for all r which just reproduces the p -wave type BCS result of Refs. 5 and 25. At finite temperatures, we allow for $\eta_r \neq 1$. The lowest energy terms correspond to a few vortex pairs ($\eta_r = -1$) which are tightly bound. We may diagonalize the Hamiltonian to determine the spectrum for these vortex patterns and find the low- T corrections to the GS results. In such a manner, we may write down a low- T series expansion for the correlation function between fermion pairs (which corresponds to string correlators in terms of the spin variables of Kitaev's model). The explicit high- and low- T series expansions will be provided elsewhere. For now, let us note that within the gapped phase, the system exhibits a finite correlation length for these string correlators.²⁵ It is clear that if a low- T series expansion about the GSs is possible, then at sufficiently low temperatures, corrections to the $T=0$ result for the string correlators can be made arbitrarily small. In particular, when such an expansion is carried out about the GSs of the $T=0$ gapped phase, the string type correlators of Ref. 12 still display a finite correlation length. [As we discussed above, that these string correlators must indeed display a finite correlation length also follows from the form of the fermionic correlator in the $\{\eta_r\}$ sector which maximizes its value; this value of the correlator provides an upper bound on the correlation function of the complete (unpinned $\{\eta_r\}$) system.] The correlation function on the right-hand side of Eq. (69) is related to a limiting form of such string correlators in which the string (defined here by the contour C_1) wraps around an entire toric cycle.

In what follows, we comment on the prospect of finite- T phase transitions in Kitaev's Honeycomb model as it follows from our expansions. As we show in the Appendixes, in the absence of a phase transition, the topological quantum error detection condition of Eq. (8) [and its equivalent Eq. (A1)], cannot be satisfied. If no transitions exist, then hardware

fault tolerance at finite temperatures may be impossible to achieve. The $T=0$ gapless phase—the phase which is not robust—seems to be amenable to a phase transition as the temperature of the system is increased. In the gapless phase, a *transition* from a system with oscillatory algebraic correlations between fermions at $T=0$ to one with exponential correlations at high temperatures is suggested by the low- and high- T expansions. For states deep inside the gapless region^{5,25} a low- T expansion (that includes sectors with a nonvanishing number of anyons) suggests that low-energy states may still carry a vanishing gap and lead to a $T>0$ phase with algebraic correlations. Thus a finite- T transition between a high- T phase with exponentially decaying correlations between fermions to a low- T *stripe type* phase with oscillatory algebraic correlations may exist.

In the absence of applied external fields which introduce a gap, the gapless sector of the theory is, however, unstable to perturbations. Of greater promise—insofar as stability to local perturbations—is the gapful region of the Kitaev model. Regrettably, no transition of the type found for the gapless phase is evident for the gapful sector of the Honeycomb model. Thus, albeit satisfying the $T=0$ condition of Eq. (5), in the absence of a finite- T phase transition, the condition of Eq. (8) cannot be satisfied at any finite temperature.

Our results on the low- and high- T expansions, which suggest that the gapful sector of the Kitaev model does not exhibit a finite- T phase transition, can be further fortified and linked to expansions carried by earlier works. These expansions elucidate how the gapful phase will not break ergodicity and cannot achieve hardware fault tolerance. The correspondence to our earlier results concerning Kitaev's toric code model is made clear by a simple mapping. In the parameter region $J_z \gg J_{x,y}$, a perturbative expansion gives the following effective Hamiltonian:⁵

$$H_{\text{eff}} \simeq -\frac{J_x^2 J_y^2}{16J_z^3} \left(\sum_s A_s + \sum_p B_p \right). \quad (74)$$

In other words, we recover Kitaev's toric code model of Eq. (20). As a result, we have as before that the logical operators ($X_{1,2}$ and $Z_{1,2}$) have a finite (system-size independent) autocorrelation time τ [Eq. (39)].

IV. THERMALLY STABLE ORDERS IN THE CASE OF HIGH-DIMENSIONAL BROKEN SYMMETRIES: A CASE OF SIMPLE NON-TQO

Some systems with nonlocal symmetries \hat{T}_μ exhibit a robustness to thermal fluctuations. These systems are natural contenders for satisfying the conditions of Eq. (8). In all of the cases that we examined, however, thermal fragility still appears.

Let us start with the simple $D=2$ orbital compass model. Its Hamiltonian is given by

$$H = - \sum_j [J_x \sigma_j^x \sigma_{j+\hat{x}}^x + J_z \sigma_j^z \sigma_{j+\hat{z}}^z], \quad (75)$$

which emulates the direction dependent interactions resulting from the anisotropy of electronic orbitals. In this model, exchange interactions involving the x component of the spin

occur only along the spatial x direction of the lattice. Similar spatial direction dependent spin exchange interactions appear for the z components of the spin. Apart from a global reflection which only appears for the isotropic point $J_x=J_z$ (and which may be broken at low T), the anisotropic orbital compass model has the following symmetry operators $\hat{O}_\mu = \prod_{j \in C_\mu} i \sigma_j^\mu$ for $\mu=x, z$. C_μ denotes any line orthogonal to the \hat{e}_μ axis. As \hat{O}_μ involves $O(L^1)$ sites, they constitute $d=1$ symmetries. The spectrum of the $D=2$ orbital compass model is gapless²⁸ in the thermodynamic limit due to the existence of the symmetries that it possesses.^{6,12,29} What is important for relaxation processes (much as was evident in the analysis of Kitaev's model) is not at all the energy spectrum but rather the free energy. It was found that the large S renditions of the isotropic ($D=2,3$) dimensional orbital compass model, although highly degenerate in the low-energy spectra in the thermodynamic limit, exhibits free energy barriers. An entropic stabilization sets in which at sufficiently low yet finite $0 < T < T^*$ leads to a sharp phase transition with a divergent relaxation time.³⁰ It is noteworthy, however, that at the isotropic point, the system is not topologically ordered as a global inversion symmetry

$$\hat{O}_{\text{reflection}} = \prod_j e^{i(\pi\sqrt{2}/4)(\sigma_j^x + \sigma_j^z)} \quad (76)$$

can be broken. It is this very symmetry which is broken below T^* . Here, the local nematic order parameter operator

$$N_j = [\sigma_j^x \sigma_{j+\hat{x}}^x - \sigma_j^z \sigma_{j+\hat{z}}^z] \quad (77)$$

attains a finite expectation value at low T ^{6,12,29,30}

$$\langle N_j \rangle_\alpha \neq \langle N_j \rangle_\beta, \quad T < T^*. \quad (78)$$

Thus measurements of this quasilocal order parameter within the GS sector lead to different answers for different contending low-energy Gibbs states (labeled by α and β). Thus N_j is not a contending logical operator—it does not satisfy Eq. (10). For all nonlocal operators \hat{O}_μ we have that at any finite T , $\langle \hat{O}_\mu \rangle = 0$ on any lattice (whether it is finite and no SSB is possible or on the infinite size lattice where the perimeter law scaling implies a vanishing $\langle \hat{O}_\mu \rangle = 0$).

Similar arguments apply for the $D=3$ orbital compass model, meaning that the $d=2$ symmetries (as well as a three-dimensional reflection symmetry) will render the TQO non-existent at finite temperatures, although the relaxation time is obviously divergent. We recently became aware of a paper by Bacon³¹ where he provides evidence (by using a mean-field approach) of a self-correcting system in a $D=3$ relative of the orbital compass model. The reason why this system is claimed to be a high-temperature self-correcting quantum memory (i.e., one without additional quantum error correction) is because, at the mean-field level, low-energy excitation energies may scale with the perimeter of the defected domain. As in the $D=3$ orbital compass model, Bacon's model displays $d=2$ (discrete) d -GLSs. The problem we see with this argument is that these symmetries can be broken giving rise to an ordered phase (i.e., one with a Landau order parameter). Then, condition (3) is going to be violated and

no TQO will be present. As in other cases in which symmetries can be broken, here Eq. (10)—a version of which is related to the weaker version [Eq. (9)] of the finite- T error detection condition of Eq. (8)—will be violated.

V. CAN THERMAL FRAGILITY BE DEFEATED?

It is clear that a physical system that is in an ordered thermodynamic phase, i.e., having a nonvanishing Landau order parameter (i.e., breaking a global symmetry), displays some sort of robustness against *local* errors due to the *collective* nature of the resulting order. This property is used to build robust *classical memories*. We have seen that topological quantum memories seem to be thermally fragile, therefore, it seems plausible that by building systems that display an order parameter we could defeat it and obtain robust *quantum memories*. In general, there seems to be a catch regarding stable quantum topological memories. According to Eq. (8), in order to ensure error detection of local perturbations at finite temperatures, the system must display at least one finite temperature phase transition (see Appendix A). We generally find that $\langle \hat{T}_\mu \rangle = 0$. As a matter of principle, even if we allow for operators $\{\hat{R}_a\}$ of a topological character (that is, nonlocal operators) which are not independent of the code's stabilizer, unless a low- T series expansion about the $T=0$ state is void, we will always get a perimeter-law type scaling for the quantum topological observables; see, e.g., Eq. (18). This expansion for local Hamiltonians [see, e.g., Ref. 16] always gives us the result that the topological observables vanish for large loops (for which local errors can be avoided at $T=0$). The expansion about $T=0$ can be void if there is a $T=0$ transition on an infinite size system. Such an occurrence matters worse once again as it implies that the robustness at $T=0$ does not obviously imply a robustness at any finite temperature (even at temperatures $T=0^+$). On any finite size system, there is no SSB and as before $\langle \hat{T}_\mu \rangle = 0$ identically. We illustrated that the same also occurs in the thermodynamic limit of these systems. Ergodicity cannot be broken insofar as the logical operators $\{\hat{T}_\mu\}$ can detect it.

We do not see an obvious way to avoid this conundrum both for the viable logical operators $\{\hat{T}_\mu\}$ which are independent of the code's stabilizer and more generally also for general nonlocal operators $\{\hat{R}_a\}$ that can encode topological operations. It should be emphasized that all of our results pertain to rigorous bounds. Although finite (and size-independent) autocorrelation times seem to be mandated, if situations exist in which $\Delta \gg k_B T$ may be realized then the autocorrelation times can be made large [see, e.g., the explicit form of Eq. (40)]. Based on heuristics and static arguments, a recent review provides estimates on these ratios in some candidate systems.³²

VI. SUMMARY

In the present article we expanded on the concept of thermal fragility and applied it to several case examples including Kitaev's honeycomb model and extensions to higher spatial dimensions. Our main results are as follows.

(i) We introduced and discussed quantum error detection at finite temperature as it applies to topologically ordered systems [Eq. (8)].

(ii) We find that a system cannot satisfy the finite temperature error detection criteria of Eq. (8) unless it displays a finite temperature phase transition. It is important to emphasize that the existence of a finite temperature transition (or several finite temperature transitions) is a necessary but not sufficient condition for Eq. (8) to hold.

(iii) In general TQO systems, whether they display finite- T phase transitions or not, and whether they have a finite size or are systems taken in the thermodynamic limit, all non-trivial topological operators may have a vanishing expectation value at all nonzero temperatures: $\langle \hat{T}_\mu \rangle = 0$. This suggests that the code might *self-correct* only up to finite cutoff times that are governed by temperature dependent effects. Depending on the relative size of the gaps to the temperature, this autocorrelation time can practically be made very large. This may go hand in hand with (system size independent) finite bounds on the autocorrelation times τ as adduced from the values of $|\langle \hat{T}_\mu(0) \hat{T}_\mu(t) \rangle|$. For times $|t| \gg \tau$, $|\langle \hat{T}_\mu(0) \hat{T}_\mu(t) \rangle|$ tends to zero [Eqs. (13), (39), (40), and (43)]. Information stored in the correlators of $\{\hat{T}_\mu\}$ may be lost. Insofar as the nonlocal *topological* operators $\{\hat{T}_\mu\}$ can detect, the system becomes ergodic at large time. Although of, apparently, less pertinence to ideal quantum memories, the same result is found for general operators $\{\hat{R}_a\}$ which encode (nonlocal) topological operations on infinite-size systems but are not necessarily independent of the code's stabilizer, Eq. (18), or correspond to symmetries.

(iv) Kitaev's toric code model can be solved exactly. At all nonzero temperatures, the autocorrelation function exhibits a finite equilibration beyond which ergodicity sets in and all information is lost. The explicit forms for the autocorrelation functions of Kitaev's toric code model are given in Eqs. (39), (40), and (43).

(v) Kitaev's honeycomb model can formally be solved at finite temperatures. The main utility of this formal solution is that it enables low- and high-temperature series expansions. These expansions suggest that this model may not be stable to thermal fluctuations [in the sense of Eq. (8)].

(vi) We showed that some candidate systems which are stable to thermal fluctuations often display local orders. These local orders render the system unstable to errors from local perturbations and the *topological protection* is lost.

A vexing question is how to quantify topological quantum error conditions at finite temperatures. The connection between rigorous bounds on quantum error detection and the more physical (although rigorous) results presented in our work is far from clear. In our opinion, this question needs to be quantitatively addressed. Given the physical results that we derived in our work, it is unclear to us if there is no way to overcome thermal fluctuations and still maintain quantum self-correction that is immune to local perturbations. We hope that our work will stimulate further studies which will address many related questions: Can $d=2$ GLSs lead to robust quantum memories? What is the relation between a finite lifetime τ and the vanishing expectation value of the

logical operators \hat{T}_μ ? What happens when a certain amount of disorder is present and thus, in general, d -GLSs are not (exactly) present? In that case it is not true that for a finite system the expectation value of the logical operators vanishes since, for example for Kitaev's toric code model $[H_K + V, Z_\mu] \neq 0$ where V represents the disorder. We conjecture that, nevertheless, in the thermodynamic limit those expectation values vanish, for example $\langle Z_\mu \rangle \rightarrow 0$, and more importantly the autocorrelation times can generally remain finite as long as V is an arbitrary quasilocal operator (and the amount of disorder is not big). Indeed, in our earlier work,⁶ we showed that the system's thermodynamic behavior and response functions may remain adiabatic even when their d -GLSs may be broken.

The main message is that we need to be careful when designing a topological quantum memory. We are forced to consider finite temperature effects since a real system is subjected to temperature effects. The common lore that a finite gap can indefinitely protect states (and thus quantum information) is not always correct. The limiting size of the autocorrelation time can often be system independent. If very low temperatures may indeed be achieved relative to the gap³² then the autocorrelation time cutoff [e.g., Eq. (40)] can be made large (albeit still finite). It may also be that systems with a well crafted *environment* can be constructed; such environments will disallow rapid thermal equilibration and thus keep the autocorrelation times high. Possible realizations of such environments can include a frequency mismatch between the natural resonant frequencies of the TQO system and the environment in which it is embedded.³³

In some of the Appendixes that follow, we elaborate on points related to the main results reported here. In Appendix A, we prove that in the absence of a phase transition (a singularity in the free energy), the finite temperature error detection condition of Eq. (8) might not hold. In Appendix B, we discuss a trivial consequence of the Lieb-Robinson bounds to *topological* quantities. In Appendix C, we sketch how rigorous bounds for the excitation of anyons may be derived in some disorder free systems in thermal equilibrium. The derivation outlined here is intimately linked to Kitaev's honeycomb model of Sec. III C. However, the result is far more general. In Appendix D, we discuss thermal equilibration time (the time for the system to become ergodic) in cases when the system displays or is close to a critical point and general scaling laws become apparent.

Recently, we became aware of the independent work of Ref. 20 which raised concerns about the reliable storage of quantum information at finite temperature. Reference 20 invoked the KMS equations in the study of Kitaev's toric code model to independently also arrive at and fortify one of the conclusions of Ref. 6 [that of the last line of Eq. (31)] from a different approach.

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APPENDIX A: TOPOLOGICAL QUANTUM ERROR DETECTION AT FINITE TEMPERATURES

We show that any system that does not exhibit a finite temperature ($T > 0$) phase transition (regardless of whether it exhibits TQO or not) necessarily violates the quantum error detection condition of Eq. (8) at *all* temperatures $T > 0$. In other words, for all $T > 0$

$$W = [\rho^{1/2} V \rho^{1/2}, \hat{T}_\mu] \neq 0. \quad (\text{A1})$$

Here, we recall that $\rho = \mathcal{Z}^{-1} \exp[-H/(k_B T)]$ with $\mathcal{Z} = \text{tr}[\exp[-\beta H]] = \exp[-\beta F]$ with F the free energy. In the absence of a phase transition, the free energy may be expanded about its high temperature (small β) limit

$$F = \sum_{k=0}^{\infty} F_k \beta^k. \quad (\text{A2})$$

All terms in the series of Eq. (A2) are c numbers. If the system displays a phase transition the expansion of Eq. (A2) has a finite radius of convergence (β_c). If the system displays no phase transition, the series of Eq. (A2) is everywhere convergent.

The proof of the assertion of Eq. (A1) for the case a divergent β_c (a system with no finite temperature transitions) is trivial. For any finite system at temperatures $T > 0$, the density matrix ρ can be expanded to all orders in $\beta = 1/(k_B T)$. We have that ($\rho = \exp[-\beta(H-F)]$)

$$W = [V, \hat{T}_\mu] - \frac{1}{2} \beta [\{(H-F_0), V\}, \hat{T}_\mu] + \frac{\beta^2}{2} [F_1, V], \hat{T}_\mu + \frac{\beta^2}{8} [\{(H-F_0), \{(H-F_0), V\}\}, \hat{T}_\mu] + O(\beta^3). \quad (\text{A3})$$

At any fixed chosen β for which $F(\beta)$ is analytic, the expansion of $\exp[-\beta(H-F)]$ in powers of β , with $F(\beta)$ held at its fixed value at the chosen β , has an infinite radius of convergence. In Eq. (A3), we collect terms of a fixed power of β when we use the expansion of Eq. (A2). In the absence of a finite temperature phase transition, the series of Eq. (A3) converges for all β . If W vanishes on a dense set of points along the real β axis then as W is an analytic function then it must also vanish everywhere in the complex β plane. All of the derivatives of W and, in particular, its zeroth order value $W_0 = [V, \hat{T}_\mu]$ must vanish. However, W_0 is independent of the system in question (it is independent of H , although $[H, \hat{T}_\mu] = 0$). For any nontrivial algebra captured by the operators $\{\hat{T}_\mu\}$, there exists quasilocal operators V that (i) have their support on the same region where $\{\hat{T}_\mu\}$ are defined, and (ii) do not commute with $\{\hat{T}_\mu\}$. Equation (A1) does not preclude a commutator which vanishes in the thermodynamic limit. The impossibility of Eq. (8) in the thermodynamic limit in all systems which exhibit SSB detectable by quasilocal operators follows from Eq. (9). In all finite size systems, there is no SSB (and the free energy is analytic— $\beta_c = \infty$) and Eq. (A1) cannot be satisfied at any nonzero temperature. A similar conclusion holds for systems which, even in their thermodynamic limit, do not exhibit $T > 0$ transitions. Ki-

taev's toric code model is precisely such a system. It is important to emphasize that the appearance of one (or more) $T > 0$ transitions is a necessary but not sufficient condition for Eq. (8) to hold.

APPENDIX B: EXTENSIONS OF THE LIEB-ROBINSON BOUND TO NONLOCAL TOPOLOGICAL QUANTITIES

Locality in quantum spin systems is often well captured by a bound on the commutator of local observables with disjoint supports. Early on, Lieb and Robinson provided estimates and bounds on precisely such commutators on nearest-neighbor spin systems on general graphs. These naturally provide bounds for the speed of propagation of quantum information. Additional works elaborated on the initial findings of Lieb and Robinson and examined extensions to several bosonic and fermionic systems.¹⁸

Stated formally, the Lieb-Robinson bound asserts that given two local operators V_a and V_b in different regions (a and b), the operator norm of the commutator in gapped systems

$$\| [V_a(0), V_b(t)] \| \leq A(\text{vol}_{\min}) \| V_a \| \| V_b \| e^{-(l-v|t|)/\xi}. \quad (\text{B1})$$

In Eq. (B1), l is the distance of the shortest path linking regions a and b (more precisely the number of links on such a path), $\text{vol}_{\min} = \min\{|a|, |b|\}$ denotes the number of vertices in the smaller of the two sets a and b . The constants A , v , and ξ depend on the maximal coordination of a given vertex of the lattice and the largest local terms $\max\{\|h_{ij}\|\}$ which appear in the argument of a nearest neighbor Hamiltonian system

$$H = \sum_{\langle ij \rangle} h_{ij}. \quad (\text{B2})$$

Similar results are found to pertain to finite temperature correlators for systems with local Hamiltonians.

The extension of Eq. (B1) to nonlocal quantities is straightforward. Let us consider the general product of local operators over an L^d -dimensional volume R_μ

$$\hat{T}_\mu = \prod_{a \in R_\mu} V_a. \quad (\text{B3})$$

If the operator norm of all V_s is bounded from above by unity then the operator norm of the commutator

$$\begin{aligned} \| [\hat{T}_1(0), \hat{T}_2(t)] \| &\leq |R_1| |R_2| \max_{a' \in R_1, b' \in R_2} \| [V_{a'}(0), V_{b'}(t)] \| \\ &\leq A |R_1| |R_2| (\text{vol}_{\min}) \| V_{a'} \| \| V_{b'} \| e^{-(l_{\min}-v|t|)/\xi}, \end{aligned} \quad (\text{B4})$$

with $a' \in R_1$ and $b' \in R_2$. In Eq. (B4), l_{\min} denotes the smallest distance between two points—one which belongs in R_1 and the other in R_2 . The first line of Eq. (B4) follows from a long-hand form of the expansion of the commutator. As \hat{T}_1 contains $|R_1|$ terms and \hat{T}_2 has $|R_2|$ terms, there are a total of $|R_1| |R_2|$ terms each containing a string of length $(|R_1| + |R_2|)$

—2) of local operators $\{V_a\}$ which bracket a single commutator between two local operators in the disjoint regions $[V_{a'}, V_{b'}]$. If the operator norm of all V_s is bounded from above by unity then the bound of the last line of Eq. (B4) follows from Eq. (B1).

As $l_{\min} \rightarrow \infty$, the equal time commutator between two distant *topological* operators vanishes. In some cases, the speed of propagation v could be related to the velocity of defect motion (e.g., domain walls).

We can similarly bound the operator norm of the commutator which appears in Eq. (8). Here, we simply have that

$$\begin{aligned} \| [\hat{T}_\mu(0), V_b(t)] \| &\leq |R_\mu| \max_{a' \in R_\mu, b} \| [V_{a'}(0), V_b(t)] \| \\ &\leq A |R_\mu| (\text{vol}_{\min}) \| V_{a'} \| \| V_b \| e^{-(l_{\min}-v|t|)/\xi}. \end{aligned} \quad (\text{B5})$$

Although the influence of events far away from the support of \hat{T}_μ (region R_μ) can become nil [as seen from Eq. (B5)], the cumulative effect of fluctuations on and proximate to R_μ can (and indeed generally does) lead to a vanishing $\langle \hat{T}_\mu \rangle$.

APPENDIX C: PROBABILITY OF OBSERVING ANYONS

We now sketch how rigorous bounds on the probability of observing anyons may be derived in one of the models that we discussed in this work (Sec. III C)—Kitaev's honeycomb model.⁵ Although some of the expressions given below are special, the result seems to be of greater applicability.

As shown by Kitaev, the GS configuration is the anyon-free configuration in which $\langle \hat{O}_h \rangle = 1$ for all plaquettes (hexagons) h . \hat{O}_h is the product of the two η_r quantities which live on the center of two vertical bonds which appear in h .²⁵ As (i) this system exhibits reflection positivity (RP)³⁴ for the Ising-type free energy (F_{Ising}) in the argument of Eq. (73),

$$e^{-\beta F_{\text{Ising}}[\{\eta_r\}]} \equiv e^{-(\beta/2)\text{tr}[M_\eta]} |\det(e^{\beta N_\eta} + 1)|^{1/2}, \quad (\text{C1})$$

(ii) the minimum of the spectrum of an $L \times L$ slab of the system (of the square lattice of Ref. 25) with open boundary conditions within a given topological sector $\{\hat{O}_h\}$ is discrete, and as (iii) within the GS the system is anyon free, there must exist a length L_c such that in all slabs of size $L > L_c$ the lowest free energy attainable over all states which are not anyon free is larger by a finite gap $\Delta_L > 0$ than the free energy within the anyon free infinite size system endowed with periodic boundary conditions. As, for example, in Ref. 35 and in particular Ref. 36, this implies that the probability of finding anyons (or vortices in Ref. 36) is exponentially suppressed at sufficiently low temperatures. This is done by tilting the lattice with all $L \times L$ blocks and noting that the probability for a given configuration is bounded from above by

$$p = \exp \left[- \frac{\beta w N_b \Delta_L}{N_c} \right], \quad (\text{C2})$$

where w is a constant of order unity, N_b is the number of *bad* blocks which contain (at least) one anyon, and N_c is the

number of $L \times L$ blocks to which a given bond is common. Combined with RP of the Ising-type Gibbs weight of Eq. (C1), we can prove that the probability of anyons is exponentially damped in β . Thus, although a gap is generally required for *quantum protection* from perturbations, it may render the probability of excitations of such anyons exponentially small. Equation (C2) is a rigorous result for any uniform RP system [including that of Eq. (64)]. It is noteworthy that the introduction of *disorder* may introduce anyons (and general defects) even at low temperatures. This has implications for impurities in quantum Hall systems.

APPENDIX D: THERMAL EQUILIBRATION TIME IN SYSTEMS WITH DISCRETE SYMMETRIES

We can use the following heuristic *metastability* argument to address the important question: How long does it take for a TQO system with d -GLSs which has been prepared in the GS (protected subspace) to equilibrate at a temperature T in the presence of d -dimensional defects? Stated alternatively, we now address the following question: How long can the system retain information before it is lost due to ergodicity? The general answer to this question is quite complex. One can use qualitative ideas borrowed from the theory of metastable states, where the thermal equilibrium state is the global minimum of the free energy F while the GS manifold is the *suitable subspace* which does not minimize F but is supposed to be locally stable. The lifetime of the protected subspace τ is clearly determined by the dynamics of the system when coupled to a thermal reservoir. The basic problem then reduces to characterizing the free energy barrier ΔF that obstructs the escape from the GS manifold. The crucial observation is that for a system of size $L \times L \cdots \times L$ for a discrete symmetry of dimension $d > 1$, the free energy barrier follows the form

$$\Delta F = aL^{d-1}\sigma \quad (\text{D1})$$

with a a positive constant and σ the tension. The point is that the presence of low-dimensional symmetries *lower* the height of the barrier because of *dimensional reduction*. The case $d=1$ was considered in Sec. III (although the energy cost was finite in that case the entropy scaled logarithmically with the system size).

Let us assume that the probability to occupy an eigenstate $|\phi_\gamma\rangle$ of H ,

$$P_\gamma(t) = \langle \phi_\gamma | U^\dagger \rho U | \phi_\gamma \rangle, \quad (\text{D2})$$

with ρ the initial state and U a unitary evolution that includes the coupling to the thermal bath, follows a master equation

$$\frac{dP_\gamma(t)}{dt} = \sum_{\gamma' \neq \gamma} [W_{\gamma\gamma'} P_{\gamma'}(t) - W_{\gamma'\gamma} P_\gamma(t)], \quad (\text{D3})$$

with transition probabilities $W_{\gamma\gamma'}$ satisfying the detailed balance condition (with equilibrium in a canonical ensemble)

$$W_{\gamma\gamma'} \exp[-\beta E_{\gamma'}] = W_{\gamma'\gamma} \exp[-\beta E_\gamma]. \quad (\text{D4})$$

It is clear that the whole complexity of the dynamical problem is in the functional form of the transition probabilities.

Let us assume that there is only one possible channel for the system to escape from the local minima.

Kitaev's model is quite special in that it displays a $T=0$ critical point. What can we say about systems that have a finite transition temperature?

Typically, in systems with a finite critical temperature, the equilibration time follows the Arrhenius form

$$\tau \sim \exp[\beta \Delta F] \quad (\text{D5})$$

with ΔF the free energy barrier for the defects. This, along with Eq. (D1), leads to the usual form

$$\tau \sim \exp[a\beta L^{d-1}\sigma]. \quad (\text{D6})$$

In what follows, we address forms of this equilibration time and the surface tension which governs it in several regimes.

1. EQUILIBRATION TIME SCALING NEAR A CRITICAL POINT

The considerations presented below apply general scaling arguments (see, e.g., Ref. 37) to general systems with discrete symmetries. For such systems, next to the critical point,

$$\sigma = k_B T_c / \xi^{d-1} \quad (\text{D7})$$

with ξ the correlation length. Near the critical temperature T_c the equilibration time scales as

$$\tau \sim \exp[c(L/\xi)^{d-1}] \quad (\text{D8})$$

with c another positive constant. We can insert the scaling of ξ with the reduced temperature $\bar{t} = (T - T_c)/T_c$ near T_c

$$\xi \sim |\bar{t}|^{-\nu} \quad (\text{D9})$$

to obtain the relaxation time near the critical point

$$\tau \sim A \exp[cL^{d-1}|\bar{t}|^{\nu(d-1)}]. \quad (\text{D10})$$

Finite surface tension when $T \rightarrow 0$

In most systems (not all) having a finite T_c , the surface tension tends to a finite value as $T \rightarrow 0$. Consequently, Eqs. (D1) and (D5) imply that the equilibration time τ is strongly divergent in both the system size L (fixed temperature) and the inverse temperature β (at fixed L).

Temperature dependent surface tension

A temperature dependent surface tension may appear in the orbital compass model and other systems which exhibit an order-out-of-disorder phenomenon.^{30,35} Here, we can prove that at sufficiently low temperatures

$$\beta\sigma \geq q > 0. \quad (\text{D11})$$

This, then, enables the proof of a finite- T phase transition by the usual Peierls argument. However, although the relaxation time τ is divergent in system size it does not, for a finite fixed L , diverge as the temperature $T \rightarrow 0$.

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$$S(K) = \{P \in P_n \mid P|\psi\rangle = |\psi\rangle \text{ for all } |\psi\rangle \in K\}.$$
 Here, P denote projection operators and P_n is the projection operator onto the n -qubit subspace K .
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$$|\langle \hat{R}_a \rangle| \sim \exp[-\kappa \text{Area}],$$
 where $\kappa > 0$ is a positive constant. In the two-dimensional ($d=2$) Ising gauge theory, an expansion about the $T=0$ state is void. Here, instead of Eq. (18), the much more dramatic falloff of this equation always holds. As before, $|\langle \hat{R}_a \rangle| \rightarrow 0$ in the limit of an infinite spatial support of \hat{R}_a . Similarly, as we showed in Ref. 6, Kitaev’s Toric code model exhibits a $T=0$ transition. In Kitaev’s Toric code model an area law of the form of this equation holds at all positive temperatures. Here, once again, $|\langle \hat{R}_a \rangle| \rightarrow 0$ in the limit of an infinite spatial support of \hat{R}_a .
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